Section 1-1, Mathematics 108

Sets

A set is a collection of objects. There can be uniformity in the collection, e.g.

{Orange, Banana, Apple}

or not

{San Francisco, Umbrella, Decimal point}

The items in brackets are called **members** of the set.

In mathematics we are usually interested in sets whose members are mathematical objects.

Examples are:

- Numbers
- Geometric objects, e.g. rectangles, triangles
- Polynomial expressions e.g. $ax^2 + bx + c$
- Sets, ... a set can also be a collection of sets

Set Notation

We can describe a set in a number of ways.

1) Using words: The set of all integers between 1 and 10

2) Using bracket notation to list the members: $\{1, 2, 3\}$

3) Using bracket notation to describe the set with an expression:

$\{x : x > 5\}$

Read this as "X such that X is greater than 5".

The characters ':' or | are both used to mean SUCH THAT

$\{x^2 : x \in \mathbb{N}\}$ Read this as "X squared such that X is a member of the set $\mathbb{N}$.

$\mathbb{N}$ is a symbol we use to mean the Natural numbers.

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\in$ is a symbol meaning "is a member of", so

$1 \in \{1, 2, 3\}$

Note that here we describe a set which is infinite. The ... is an ellipsis which means continue on indefinitely.

Note: $\{\{1, 2\}, \{3, 4\}\}$ is a set with two members. Each member of the set is a set with two members.

We've already seen that a set can have a finite number of members or an infinite number:

$\{1, 2, 3\}$ - Finite
$\{1, 2, 3, \ldots\}$ - Infinite
$\{x : 0 \le x \le 1\}$ - Infinite

A set can also have no members. This is called the empty or null set. We can write the null set as either

$\{\ \}$ or $\varnothing$

Note: $\{\varnothing\}$ is not the empty set. It is a set with one member, the empty set.

Operations on Sets

Union

There are a number of standard operations we can perform on sets.

The union of two sets is a set with all the members of the two sets.

$$\{1,2,3\}\bigcup\{3,4,5\}=\{1,2,3,4,5\}$$

The symbol we use looks like a U so it is easy to remember.

Note that the two sets each have the member 3, but it is represented only once in the union. Each member of a set must be unique.

Intersection

The intersection of two sets is a set with only the members that are in both sets.

$$\{1,2,3\}\cap\{3,4,5\}=\{3\}$$

$$\{1,2,3\}\cap\{4,5\}=\varnothing$$

$$\{1,2,3\}\cap\{1,2,3\}=\{1,2,3\}$$

Sometimes we will represent sets with letters, the same way we use variables in mathematical expressions. Usually we use a capital letter to represent a set, for example:

$$A=\{1,2,3\}$$

Using a letter as a variable we can write some simple laws of sets, for example:

$$A=A\bigcup A$$
$$A=A\bigcap A$$

Subsets

A subset of a set A is a set B whose members are all in the original set.

The subsets of $A = \{1, 2\}$ are

$\{\ \}, \{1\}, \{2\}, \{1, 2\}$

Note that the empty set and the set itself are always subsets of a set.
The set and the empty set are called **improper** subsets and all other subsets are called **proper** subsets.

We can write A is a subset of B using the symbols $\subset, \supset, \subseteq, \supseteq$

$$A \subseteq B$$
$$B \supseteq A$$
$$A \subset B$$
$$B \supset A$$

If you think of the $\subset$ as a < you can see which set can have more members.

Sometimes $\subset$ *and* $\supset$ mean a proper subset.

The set that includes all subsets is called the **power** subset. It can be written this way

$$A = \{1, 2\}$$
$$P(A) = \{\{\ \}, \{1\}, \{2\}, \{1, 2\}\}$$

Note that the power set of a set with 2 elements has 4 elements. Also

$$B = \{1, 2, 3\}$$
$$P(B) = \{\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

A set with 3 elements has a power set with 8 elements.

This suggests that a set with *n* members has 2^n members.

Complement of a Set

The complement of a set A is a set B with all the elements that are not in A.

We write this

$A^c = B$

Note that we need to know what elements are being considered. These elements are a set we call U or the **universe** of elements.

If $U = \{1, 2, 3, 4\}$ and $A = \{1, 2\}$ then $A^c = \{3, 4\}$

Note that if U=A the complement of A is the null set.

Also note that the complement of the null set is always the universe.

Important sets and their symbols

We give these important sets the following symbols:

The natural numbers - $\mathbb{N} = \{1, 2, 3, \cdots\}$

The integers - $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ The Z stands for zahlen which is the German word for numbers.

The rational numbers - $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\}$ The Q stands for quotient

The real numbers - $\mathbb{R}$

The complex numbers - $\mathbb{C} = \left\{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\right\}$

Note that: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Properties of the real numbers

We will list some of the properties of addition an multiplication of the real numbers.

Addition

Closure property	if $x, y \in \mathbb{R}$ then $x+y \in \mathbb{R}$
Associate property	if $x, y, z \in \mathbb{R}$ then $(x+y)+z = x+(y+z)$
Commutative property	if $x, y \in \mathbb{R}$ then $x+y = y+x$
Identity property (of 0)	if $x \in \mathbb{R}$ then $x+0 = 0+x = x$
Inverse property	if $x \in \mathbb{R}$ then there exists $-x$ such that $x+-x=0$

Multiplication

Closure property	if $x, y \in \mathbb{R}$ then $xy \in \mathbb{R}$
Associate property	if $x, y, z \in \mathbb{R}$ then $(xy)z = x(yz)$
Commutative property	if $x, y \in \mathbb{R}$ then $xy = yx$
Identity property (of 1)	if $x \in \mathbb{R}$ then $x \cdot 1 = 1 \cdot x = x$
Inverse property	if $x \in \mathbb{R}$ *and* $x \neq 0$ then there exists $\frac{1}{x}$ such that for all $x \cdot \frac{1}{x} = 1$

Distributive property (of multiplication over addition)

$x, y, z \in \mathbb{R}$ then $x(y+z) = xy + xz$

Note that subtraction and division do not share all of these properties.
Also note that $\mathbb{Q}$ *and* $\mathbb{C}$ have all of these properties, but $\mathbb{N}$ does not.

Intervals

On important type of subset of the reals is called an **interval**.
An interval is a continuous set of numbers.
You can think of it as a line segment on the real number line.

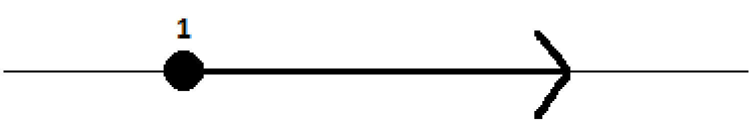
This describes a **closed** interval $\{x : 1 \le x \le 2\}$ which is written [1,2].

This describes a **open** interval $\{x : 1 < x < 2\}$ which is written (1,2).

Note that this is not an ordered pair, or point on an *x*,*y* graph.

An interval can also be half open $\{x : 1 \le x < 2\}$

One end of the interval can be infinite. $\{x : x \ge 1\}$



We write this as [1,∞). Note that the infinite side is considered open, not closed.

Finally we can write all the real numbers as an interval $\mathbb{R} = (-\infty, \infty)$

Absolute Value

The absolute value is defined as follows

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Some properties of the absolute value

$$|x| \geq 0$$
$$|x| = |-x|$$
$$|x - y| = |y - x|$$
$$|xy| = |x||y|$$
$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

These should all be obvious

Finally, we have the triangle inequality

$$|x + y| \leq |x| + |y|$$

Note that if *x* and *y* are both positive or both negative, you get equality and otherwise you get <.

Distance on a line

Let $d(x, y)$ be the distance between two points on the number line, then

$$d(x, y) = |x - y|$$